CS 712 Dissipative Systems Theory

CS 712 Fall 2017

September 14, 2018

Chapter 1

Dissipative Systems Theory

1.1 Introduction

Dissipative dynamical systems are special cases of Dynamical system where a fundamental constraint is placed on their behavior in terms of 'dissipative inequality'. Most common examples of dissipative systems are electrical networks in which part of the electrical energy is dissipated in the resistors in the form of heat, viscoelastic systems in which viscous friction is responsible for a similar loss in energy, and thermo-dynamical systems for which the second law postulates a form of dissipation leading to an increase in entropy. One important aspect of studying these system is for analyzing the stability of control systems. One of the main resulKYPts in stability theory involving dissipative dynamical feedback system containing passive dynamical system in both the forward and the feed-back loop is itself passive and thus stable.

In order to understand the principles of dissipative system theory, we first need to understand the concepts of Lyapunov functions, passivity, supply rate and storage functions. In the below sections, we provide an intuitive explanation of the required concepts to properly understand the principles of dissipative systems and dive deep into storage functions.

1.2 Lyapunov Functions

Generally, *Lyapunov functions* are real-valued functions of system's state which are monotonically non-increasing on every signal.

Definition 1 A **Lyapunov function** is a scalar function $L(y)$ defined on a region D that is continuous, positive definite, $L(y) > 0$, $\forall y \neq 0$, and has continuous first-order partial derivatives at every point of D. The derivative of L with respect to the system $y' = f(y)$, written as $L^*(y)$ is defined as $L^*(y) = \nabla L(y).f(y)$

In simple terms, for a simple autonomous system $y = f(x)$ where $f(x) = -x$ and $y = \dot{x}$ i.e $\dot{x} = -x$, there can be many functions which are monotonically non-increasing. Let $V(x) = x^2$ be a function. This function is always positive for every value of x. Then,

$$V^*(x) = \nabla V(x).f(x) \tag{1.1}$$

$$V^*(x) = 2x. - x \tag{1.2}$$

$$V^*(x) = -2x^2 \tag{1.3}$$

Since, function $V^*(x)$ is always decreasing with $x \in \mathbb{R}$, $V(x)$ is a Lyapunov function. We can consider another function $W(x) = x^4$. This function is also always positive for every value of x. Then,

$$W^*(x) = \nabla W(x).f(x) \tag{1.4}$$

$$W^*(x) = 3x^3. - x \tag{1.5}$$

$$W^*(x) = -3x^4 \tag{1.6}$$

Since, function $W^*(x)$ is always decreasing with $x \in \mathbb{R}$, $W(x)$ is a Lyapunov function.

From above example, we can see that for a give system, there can be many Lyapunov function. Wolfram [1] has another good example on Lyapunov function.

For a conservative system, the total energy is always a Lyapunov function. The Lyapunov function have an explicit upper bound imposed on their increments along system trajectories i.e $L^*(x, t_1) - L^*(x, t_0) \leq 0$, $\forall t_1 \geq t_0 \geq 0$

As seen in Figure 1.1, the Lyapunov function acts as a wrapper for the trajectory followed by the system. We can say that Lyapunov function is a bounding function to the system's trajectory which is non-increasing with respect to time.

1.3 Dissipative System

Dissipative systems [2] are dynamical systems with states $x(t)$, inputs $u(t)$ and outputs $y(t)$, which satisfy the so-called "dissipation inequality". A system is said to be dissipative if there exist a continuous non-negative function $V(x)$ of the real variable x, called the storage function, such that the following inequality, known as the dissipation inequality, always holds:

$$\frac{dV(x(t))}{dt} \leq u(t).y(t) \tag{1.7}$$

The function $u.y$, where . denotes the scalar product, is called the "supply rate". The physical interpretation is that $V(x)$ is the energy stored in the system, where

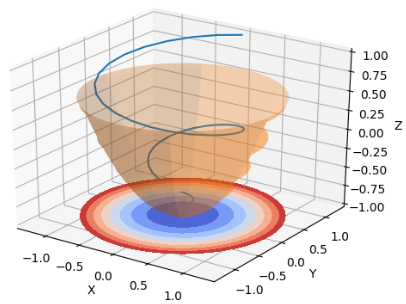

Figure 1.1 Contour of Lyapunov function. As time passes, the trajectory represented by blue curve converges for the convex Lyapunov function.

$u.y$ is the energy that is supplied to the system. We can think of a dissipative system as a system that absorbs energy and storage function as a function that models something like the power delivered to the system when the input value is u and output value is y. The dissipative system which holds the inequality in equation 1.7 is knows as conservative or passive system. A simple dissipative system is show in Figure 1.2. Electric circuits are good example of dissipative system where V (Voltage) and I (current) is input and storage function is the energy stored in capacitors and inductors.

The study of dissipative system is important as its study relates to the study of stability of the system not only limited to linear systems. As simple check for the stability of the system is Bounded Input/Output method. If both input and output are bounded in a system then that system is stable. Taking this concept a little further, lets consider the same bounded system in terms of energy. If the system cannot produce energy more than it has been supplied than that system is passive i.e., a system is called passive if it contains no energy sources and cannot put out more energy than has been put into it. Intuitively, if the system output is always less than the energy put into it, it almost always remains in proximity of stable region. Thus in most cases, a passive system implies stable system. The most fascinating result related to passivity and stability is that if we connect a number of passive system together, the overall system is still passive. This powerful results enables us to check the stability of a complicated system effectively.

Let's take an example of water tank system as show in Figure 1.3.

This example is taken from Dissipative and Passivity [3]. Let the input be the inlet flow rate $u = F_i(t)$, the state variable is the liquid level $x(t)$ and the output variable is the liquid pressure $y = p(t) = \rho g x(t)$. Suppose that the outlet is flowing under the influence of gravity, i.e.,

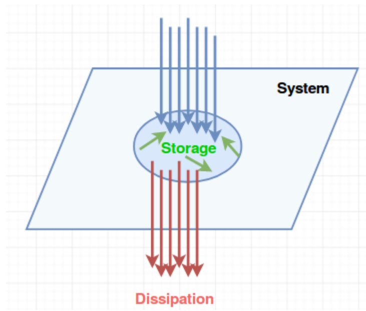

Figure 1.2 Simple representation of dissipative systems

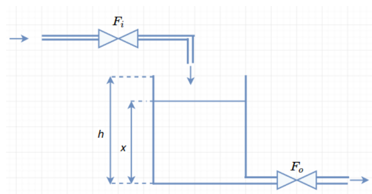

Figure 1.3 A gravity tank system

$$F_o(t) = C_v\sqrt{x(t)} \tag{1.8}$$

where C_v denotes the value coefficient and F_o is the mass flow rate. The mass balance is given by

$$\rho A \frac{dx(t)}{dt} = \rho F_i(t) - \rho F_o(t) = \rho F_i(t) - \rho C_v\sqrt{x(t)}, \tag{1.9}$$

leading to

$$\frac{dx(t)}{dt} = -\frac{C_v}{A}\sqrt{x(t)} + \frac{1}{A}F_i(t), y(t) = p(t) = \rho g x(t), \tag{1.10}$$

where A is the cross-sectional area of the tank and ρ is the density of the liquid. Denote the mass in the tank as m. Half of the potential energy stored in the tank is given by the following equation:

$$S(t) = S(x(t)) = \frac{1}{2}m(t)gx(t) = \frac{1}{2}[\rho A x(t)]gx(t) = \frac{1}{2}A\rho g x^2(t). \tag{1.11}$$

The inlet flow into the system increase the potential energy in the tank. The increment of potential energy per unit time can be represented by a function of the input and output:

$$w(t) = y(t)u(t) = \rho g F_i(t)x(t) \tag{1.12}$$

The rate of change of the potential energy is given by taking the derivative along the trajectory of x(t):

$$\frac{dS(t)}{dt} = \frac{\partial S}{\partial x}\frac{dx}{dt} = A\rho g x(t)\big[\frac{1}{A}\big(F_i(t) - C_v\sqrt{x(t)}\big)\big] \tag{1.13}$$

$$= -C_v \rho g x(t)\sqrt{x(t)} + \rho g F_i(t) x(t) < w(t) \tag{1.14}$$

Note that in the range of definition of x, the first term of 1.14 is always negative. Therefore the rate of change of the stored energy in the tank is less than that supplied to it by the inlet flow rate ($w(t)$). As such, the tank system "dissipates" its potential energy through both the inlet flow (F_i) and the liquid pressure p, which is a function of both the input and output. This is called a dissipative system. Because the potential energy $S(t)$ is a positive definite function of the state variable $x(t)$, it can be treated as a Lyapunov function. When $F_i(t) = 0$,

$$\frac{dS(t)}{dt} < 0, \forall x \neq 0 \tag{1.15}$$

Therefore, the equilibrium $x = 0$ is asymptotically stable (AS). If the outlet valve is completely shut off (i.e., $C_v = 0$), then the energy flow into the tank is totally stored. In this case, this process becomes lossless and the equilibrium $x = 0$ is stable. Here the change of the potential energy is the storage function.

Now we will formally define Dissipative Dynamical Systems and give example of interconnected systems and then discuss in details about storage functions.

All the notation and definition are taken from Dissipative Dynamical Systems 1: General Theory [2]. We found it to be the excellent resource for Dissipative Dynamical Systems.

Assume that a dynamical system Σ is given together with a real-valued function w defined on $U \times Y$. This function is called *supply rate*. We assume that for any $(t_1, t_0) \in R_2^+, u \in U$, and $y \in Y$, the function $w(t) = w(u(t), y(t))$ satisfies $\int_{t_o}^{t_1} |w(t)|dt < \infty$, i.e., w is locally integrable.

Definition 2 A dynamical system Σ with supply rate w is said to be *dissipative* if there exists a nonnegative function $S : X \to R^+$, called the *storage function*, such that for all $(t_1, t_0) \in R_2^+, x_0 \in X$, and $u \in U$,

$S(x_0) + \int_{t_0}^{t_1} w(t)dt \geq S(x1)$

where $x_1 = \phi(t_1, t_0, x_0, u)$ and $w(t) = w(u(t), y(t))$, with $y = y(t_o, x_o, u)$.
The above inequality is called the 'dissipative inequality'.

Definition 3 Along with storage function, *available storage* is also important and it is the maximum amount of storage which may at have been extracted from a dynamical system. *Available storage* is the largest amount of energy that can be extracted from the system give the initial condition $x(0) = x$. The *available storage*, S_a, of a dynamical system Σ with supply rate is the function from X into R^e defined by

$S_a(x) = \sup_{x \to t_1 \geq 0} - \int_0^{t1} w(t)dt$

where the notation $x \rightarrow$ denotes the supremum over all motions starting in state x at time 0 and where the supremum is taken over all $u \in U$. The Dissipative Dynamical System [2] have more detailed explanation and theorems on *available storage*. The essential point is that *available storage* is an important in determining whether or not a system is dissipative. The concept of *required supply* is also important in dissipative systems. *required supply* assumes that there exists a point $x^* \in X$ such that $S(x^*) = \min_{x^* \in X} S(x)$. To define $S_r(x)$, we consider a sequence of states x_n with $\lim_{n\rightarrow\infty} S(x_n) = \inf_{x\in X} S(x)$ and define $S_r(x) = \lim_{n\rightarrow\infty} S_{r,n}(x)$

Theorem 1 [2] provides the proof that the available storage , S_a, is finite for all $x \in X$ if and only if Σ is dissipative. Moreover, $0 <= S_a <= S$ for dissipative dynamical systems and S_a is itself a possible storage function. The storage function of a dissipative dynamical system satisfies *a prior* inequality $S_a \leq S \leq S_r$, i.e., a dissipative system can supply to the outside only a fraction of what it has stored and can store only a fraction of what has been supplied to it. But not every function bounded by this *a priori* inequality will be a possible storage function.

We have left out the definition and theorems related to *required supply*, *reachable*, *controllable* and others as they are detailed in [2].

1.4 Storage Functions

In simple terms, a storage function gives the amount of energy stored inside the system at any instant of time and is dependent on the system states. Formally, storage function are generalization of Lyapunov Function. In equation 1.7, V(x(t)) is a storage function but when $u(t).y(t) = 0$ then it becomes a Lyapunov function. Thus, the main importance of storage function in case of dissipative systems is similar to the concept of Lyapunov stability for closed systems. To clarify, open systems are those system which as able to exchange matters with its environment where are closed systems are those system which are isolated from their environment.

The storage function of a dissipative dynamical system satisfies the *a prior* inequality $S_a \leq S \leq S_r$, i.e., a dissipative system can supply to the outside only a fraction of what it has stored and can store only a fraction of what has been supplied to it. But not every function bounded by this *a priori* inequality will be a possible storage function. This *inequality* is the main property of storage function. The next most important properties of storage function is convexity.

Theorem 1 *The set of possible storage functions of a dissipative dynamical system forms a convex set. Hence* $\alpha S_a + (1-\lambda)S_r, 0 \leq \lambda \leq 1$, *is a possible storage function of a dissipative dynamical system whose state space is reachable from* x^*.

The proof for this theorem is on Page 11 of Dissipative Dynamical System [2].

The consideration of the storage function is an extremely useful tool in stability investigations and by properly choosing the supply rates one may indeed obtain an interpretation for most of the existing stability criteria. In constructing a storage function it is natural to proceed to the evaluation of either the available storage or the required supply. These however lead to variational problems and it is only in exceptional circumstances that one may solve such problems, particularly if the dynamical system Σ is nonlinear. The concept of interconnected systems becomes in fact very useful in this context: it allows one to construct storage functions which correspond to neither the available storage nor the required supply, and which may be constructed by solving variational problems. Details on interconnected systems are available in Dissipative Systems Theory [2].

Given a dissipative dynamical system and the supply rate, does there exists a storage function V such that the dissipation inequality holds? If the dissipation inequality holds, is the storage function unique? How to design storage functions for a system? We will try to answer all these questions in consequent sections.

Before we go more deeper into Storage functions, we need to formally define passive system with respect to its storage function

Definition 4 Passive system. A system is said to be passive if it is dissipative with respect to the following supply rate:

$$w(u(t), y(t)) = u^T(t)y(t), \tag{1.16}$$

and the storage functions $S(x)$ satisfies $S(0) = 0$

Definition 5 Lossless system. A passive system Σ with storage function $S(x)$ is said to be lossless if for all $t_1 \geq t_0 \geq 0, x_0 \in X$ and $u \in U$,

$$S(x) - S(x_0) = \int_{t_0}^{t_1} y^T(t)u(t)dt. \tag{1.17}$$

Definition 6 Strictly passive system. A passive system Σ with storage function $S(x)$ is said to be strictly passive if there exists a positive definite function $V : X \to \mathbb{R}^+$ such that for all $t_1 \geq t_0 \geq 0, x_0 \in X$ and $u \in U$,

$$S(x) - S(x_0) = \int_{t_0}^{t_1} y^T(t)u(t)dt - \int_{t_0}^{t_1} V(x(t))dt. \tag{1.18}$$

In the previous example of water tank system, the storage function is the total potential energy stored in the tank system, given by 1.11. The supply rate give by 1.12 is the inner product of the input and output. Therefore, the tank system is state strictly passive when the outlet valve is open and is lossless when the outlet value is closed.

The construction of storage functions is very well understood, particularly for finite dimensional linear systems and quadratic supply rates and leads to the

KYP(Kalman-Yacubovich-Popov), Linear Matrix Inequality(LMI), ARIneq, Algebraic Riccati Equation (ARE), semi-definite programming, spectral factorization, Lyapunov functions, robust control, positive and bounded real functions and more. The storage function for conservative or passive system is unique but for other systems it is far from unique. An unique storage function can obtained for a dissipative system if the system is assumed to be formed by interconnection of smaller dissipative systems. There are two 'canonical' storage functions: the *available storage* and the *required supply* which we have described in definition 3.

We will briefly introduce inter-connected system [2] here.

1.5 Interconnected Systems

The storage function is only uniquely determined by the dissipation inequality if and only if the required supply equals the available storage. In other cases, the dissipation inequality only provides bounds for storage function. But in case of interconnected dissipative sub-systems, the valid storage functions are few. The interconnected dissipative sub-system is quite similar to Linear Fractional Transformation representation of Sub-system structure. In this case, each subsystem should satisfy the property of dissipative system. If the interconnection of the sub-system is neutral and is lossless than the storage function of that system can be uniquely determined. Details on Theorems and definitional of Interconnected system is described here [2].

Theorem 2 *An equilibrium point* $x^* \in X$ *of a dissipative dynamical system* Σ *is stable if the storage function* S *is continuous and attains a strong local minimum at* x^**. Moreover* S *is a Lyapunov function in the neighborhood of* x^*

If storage function of each sub-system of dissipative system is Stable (passive) then the complete system with neutral interconnected is also stable. The construction of storage function can be started with available storage or required supply but it can lead to various problems and we might end up with equation which are very hard to solve. In this exception cases where the design of storage function is complex, the concept of interconnected system is very important.

1.6 Example

Lets consider a generic dynamical systems Σ_1 and Σ_2 and assume that $U_1 = U_2 = Y_1 = Y_2$ are inner production spaces. Assume now that Σ_1 and Σ_2 are interconnected via the constraint $u_2 = y_1$ and $u_1 = -y_2$. This results in the feedback system as shown in figure 1.4.

Lets study the stability of this system based on the discussion made so far on dissipative systems. Assume that we associate the supply rate $w_1(u_1, y_1)$ with Σ_1 and the supply rate $w_2(u_2, y_2)$ with Σ_2. If w_1 and w_2 are such that $w_1(u, y) + w_2(y, -u) = 0 \forall u, y$, then the feedback system may be considered

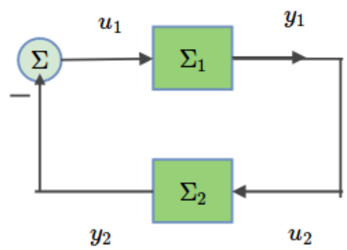

Figure 1.4 Dissipative Feedback system

as an interconnected system with the neutral interconnection constraint: $u_2 = y_1, u_1 = -y_2$. Neutral interconnection is the first requirement for the interconnected system to be stable. Now to show that this system is stable we need to show that the sub-system Σ_1 is dissipative with respect to w_1 and that Σ_2 is dissipative with respect to w_2. We can use different types of supply rates but $w_1 = ||u_1||^2 - ||y_1||^2, w_2 = ||u_2||^2 - ||y_2||^2$ is quite popular and is known as quadratic supply rate.

For this example, lets consider a specific non-linear system Σ_1 represented by below PDE

$$\dot{x_1} = -x_1 \tag{1.19}$$

$$y_1 = x_1$$

and Σ_2 represented by

$$\dot{x_2} = \frac{1}{1 + e^{x_2}} \tag{1.20}$$

$$y_2 = x_2$$

If we consider a quadratic supply rate for these system then, $w_1 = ||x_2||^2 - ||x_1||^2$ and $w_2 = ||x_1||^2 - ||x_2||^2$. The stability criteria [2] requires us to prove that the system Σ_1 and Σ_2 are dissipative with respect to supply rates w_1 and w_2 respectively in order to prove the whole system is stable. Lets consider a function $S_1(x_1) = x_1^2$ for system Σ_1 and $S_2(x_2) = \frac{1}{1+e^{x_2}}$. Both the function can be called storage function first, if we show that these function are Lyapunov functions and second holds the dissipative inequality.

For show the first part,

$$S_1^*(x_1) = \Delta S_1(x_1). - x_1$$

$$S_1^*(x_1) = -2x_1^2$$

Since the function $S_1^*(x_1)$ is always decreasing with $x_1 \in \mathbb{R}$, $S_1^*(x_1)$ is a Lyapunov function.

Similarly,

$$S_2^*(x_2) = \Delta S_2(x_2).\frac{1}{1+e^{x_2}}$$

$$S_2^*(x_2) = -\frac{1}{e^{2x_2}}.\frac{1}{1+e^{x_2}}$$

Since the function $S_2^*(x_2)$ is always decreasing with $x_2 \in \mathbb{R}$, $S_2^*(x_2)$ is a Lyapunov function.

For second part, since, $S_1(x_1) \le w_1 \; \forall x_1, x_2 \in \mathbb{R}$ and $S_2(x_2) \le w_2 \; \forall x_1, x_2 \in \mathbb{R}$ it holds the dissipation equality and thus both the system Σ_1 and Σ_2 are dissipative systems. From the interconnected systems theorem [2], neutral interconnected dissipative system are also stable. Thus the overall system is stable and overall flow rate of the system is $w_1 + w_2$ and storage function is $S_1(x_1) + S_2(x_2)$

The popular methods for finding storage function for dissipative system are KYP property and LMI. We will study these two in details.

1.7 Kalman-Yacubovich-Popov Property

Definition 7 Kalman-Yacubovich-Popov property. [3] Consider a system

$$H : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1.21}$$

where $x \in \mathbf{X} \subset \mathbb{R}^n$, $u \in \mathbf{U} \subset \mathbb{R}^m$ and $y \in \mathbf{Y} \subset \mathbb{R}^m$. It is said to have the Kalman-Yacubovitch-Popov (KYP) property if there exists a $\mathbb{C}^1$ nonnegative function $S(x) : \mathbb{X} \to \mathbb{R}^+$, with $S(0) = 0$ such that

$$L_f S(x) = \frac{\partial S(x)}{\partial x} f(x) \le 0, \tag{1.22}$$

$$L_g S(x) = \frac{\partial S(x)}{\partial x} g(x) = h^T(x), \tag{1.23}$$

for each $x \in X$.

The term $L_f S(x) = \frac{\partial S(x)}{\partial x} f(x)$ is called the *Lie derivative*, which is defined as follows

Definition 8 Lie Derivative. Given a C^1 nonlinear scalar function $S(x) : \mathbb{R}^n \to \mathbb{R}$ and a vector function:

$$f(x) = [f_1(x), f_2(x), \cdots, f_n(x)]^T \in \mathbb{R}^n \to \mathbb{R}^n \tag{1.24}$$

on a common domain $X \subset \mathbb{R}^n$. The derivative of $S(x)$ along f is defined as

$$L_f S(x) = \frac{\partial S(x)}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial S(x)}{\partial x_i} f_i(x). \tag{1.25}$$

The repeated Lie derivative is defined as

$$L_f^k S(x) = \frac{\partial\big(L_f^{k-1}S(x)\big)}{\partial x} f(x), \tag{1.26}$$

with $L_f^0 S(x) = S(x)$

The above definition doesn't make intuitive sense of what Lie derivative is. In simple terms, the Lie derivative is just taking the derivative of a tensor by a vector. Lets see a simple example of Lie derivative.

Example 1 Let $S(x) = x^3$ and $f_(x, y) = (x^2, 2y^2)$ Then

$$L_f S(x) = \sum_{i=1}^{n} \frac{\partial S(x)}{\partial x_i} f_i(x) = x^2 \frac{\partial f}{\partial x} + 2y^2 \frac{\partial f}{\partial y} = x^2(3x^2) + 2y^2(0) = 3x^4$$

Theorem 3 *A system H which has the KYP property is passive, with a storage function $S(x)$. Conversely, a passive system having a C^1 storage function has the KYP property.*

For the water tank example, $f(x) = -\frac{C_v}{A}\sqrt{x}, g(x) = \frac{1}{A}$ and $h(x) = \rho g x(t)$. With the storage function $S(x) = \frac{1}{2}A\rho g x^2$, it is easy to very that the tank system has the KYP property.

$$L_f S(x) = \frac{\partial S(x)}{\partial x} f(x) = A\rho g x\big(-\frac{C_v}{A}\sqrt{x}\big) = -/rhogC_v x(t)\sqrt{x(t)} \tag{1.27}$$

Since, $x(t) \leq 0$, $L_f S(x) \leq 0$

$$L_g S(x) = \frac{\partial S(x)}{\partial x} f(x) = A\rho g x(t)\frac{1}{A} = \rho g x(t) = h(x) \tag{1.28}$$

Thus, for the water tank system with storage function $S(x)$, KYP property is satisfied.

For a linear time invariant (LTI) system, there exits a quadratic storage function $S(x) = x^T P x$ (with a positive definite matrix P), leading to the following linear version of the KYP condition.

Theorem 4 *Consider a stable LTI system*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1.29}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$. This system is passive if and only if there exist matrices $P, L \in \mathbb{R}^{n\times n}$,$Q \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ with $p > 0, l > 0$ such that

$$\begin{aligned} A^T P + PA &= -A^T Q - L, \\ B^T P - C &= -W^T Q, \\ W^T W &= D + D^T. \end{aligned} \tag{1.30}$$

For the systems with relative degree 0, the above condition can be represented using a linear matrix inequality (LMI), which is often referred to as the positive-real lemma.

Theorem 5 *A stable LTI system with $D \neq 0$ is passive if and only if there exists a positive definite matrix P such that:*

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D - D^T \end{bmatrix} < 0 \tag{1.31}$$

When $D = 0$, the above condition is reduced to

$$\begin{aligned} A^T P + PA &< 0, \\ B^T P &= C. \end{aligned} \tag{1.32}$$

The above equation are the linear version of KYP conditions.

1.8 Example Problem

The system described by $\dot{x} = f(x, u), y = g(x), x(0) = 0$, with $u(t), y(t) \in \mathbb{R}^m$, is said to be *passive* if

$$\int_0^t u(\tau)^T y(\tau) d\tau \geq 0 \tag{1.33}$$

holds for all trajectories of the system, and for all t [4].

(a. Establish the following Lyapunov condition for passivity: If there exists a function V such that $V(z) \geq 0 \forall z, V(0) = 0$, and $\dot{V}(z, w) \leq w^T g(z) \forall w, z$, then the system is passive

(b. Now suppose the system is $\dot{x} = Ax + Bu, y = Cx$, and consider the quadratic Lyapunov function $V(z) = z^T P z$. Express the conditions found in part (a) as a matrix inequality involving A, B, C and P.

(c. Now consider the specific case with A $= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -5 & -2 \end{bmatrix}$, B $= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, C $= \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}$. Use an LMI solver to find a matrix P for which Lyapunov function $V(z) = z^T P z$ establishes passivity of the system.

Solution:

(a. We need to show that if Lyapunov passive function V holds $V(z) \geq 0 \forall z, V(0) = 0$, and $\dot{V}(z, w) \leq w^T g(z) \forall w, z$, then the system represented by the Lyapunov function is passive system i.e

$$\int_0^t u(\tau) y(\tau) d\tau = \int_0^t u(\tau)^T g(x(\tau)) d\tau \geq 0,$$

Then

$$\int_0^t u(\tau)^T g(x(\tau))d\tau \geq \int_0^t \dot{V}(x(\tau), u(\tau))d\tau$$

Expanding the integration of the right hand term

$$V(x(t)) - V(0)$$

Since, $V(z) \geq 0$ and $V(0) = 0$, $V(x(t)) - V(0) \geq 0$, the given conditions for a Lyapunov function leads to passive system, thus the system is passive.

(b. Using a Lyapunov function of the form $V(z) = z^T P z$, the passivity condition of part (a) is satisfied if we find a matrix $P \leq 0$ such that $\dot{V}(z, w) = \dot{z}^T P z + z^T P \dot{z} \leq w^T C z$ holds for all w, z. The condition is also know as KYP lemma. The restricted version of KYP lemma when $D = 0$ is know as Linear Matrix Inequality which we have discussed in previous section. The conditions stated in part (a) can be restarted as: there exists a $P \leq 0$ such that, for all w and z,

$$\begin{aligned} &\dot{z}^T P z + z^T P \dot{z} \\ &= (Az + Bw)^T P z + z^T P (Az + Bw) \\ &= z^T (A^T P + PA) z + w^T B^T P z + z^T P B w \\ &\leq w^T C z, \end{aligned}$$

The above conditions can be represented concisely in matrix form as:

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D - D^T \end{bmatrix} < 0$$

For above condition, if $D = 0$, then

$$\begin{aligned} A^T P + PA &< 0, \\ B^T P &= C. \end{aligned}$$

Thus the equivalent conditions found on part (a) in terms of A, B, C, and P is shown above and is known as LMI.

(c. Given matrix A, B, C we can find the value of P for the linear matrix inequality represented by KYP lemma using semidefinite programs [5]. Using the SDP specification into a Matlab script, we find

$$P = \begin{bmatrix} 2.0197 & 0.8472 & 0.1528 \\ 0.8472 & 1.7439 & 0.2561 \\ 0.1528 & 0.2561 & 0.2439 \end{bmatrix}$$

Bibliography

[1] E. W. Weisstein, "Lyapunov function at."

[2] J. C. Willems, "Dissipative dynamical systems part i: General theory," *Archive for rational mechanics and analysis*, vol. 45, no. 5, pp. 321–351, 1972.

[3] J. Bao and P. L. Lee, "Dissipativity and passivity," *Process Control: The Passive Systems Approach*, pp. 5–41, 2007.

[4] E. P. S. Boyd, "Ee363 homework 8 solutions."

[5] ——, "Ee363 review session 4: linear matrix inequalities."

[6] J. C. Willems, "Dissipative dynamical systems," *European Journal of Control*, vol. 13, no. 2-3, pp. 134–151, 2007.